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Energy-Momentum-Stress Tensor and Gravitational Field of Uniformly Rotating Masses†

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Abstract

Beginning with a general ansatz for the energy-momentum-stress tensor of masses in uniform rotation we determine uniquely the energy tensor and the gravitational field of two particular systems; namely, a thin, rotating, hollow cylinder and a spinning rod (dumb-bell). All calculations are made within the framework of linearised theory, but no restriction is made upon angular velocity, except that given by the velocity of light.

1. *Introduction*

Soon after the formulation of Einstein's theory of general relativity, Thirring (1918, 1921) made use of the linearised field equations in order to investigate gravitational effects of a uniformly rotating shell (Thirring effect). In another paper Lense & Thirring (1918) studied the influence of rotating planets upon the motion of their moons. The main result of these papers is the reduction of gravitational effects to 'centrifugal' and 'coriolis' forces. In his ansatz for the energy tensor Thirring did not take into consideration the interactions of matter (stresses) which enforce the rotation. For this reason his calculations were revised by Bass & Pirani (1956) and independently by Hönl & Maue (1956). Recently, Brill & Cohen (1965) —and especially Teyssandier (1970)—have developed a method of successive approximation with respect to angular velocity ω, beginning with a nonrotating spherical base metric.‡ However, all these papers still restrict to small velocities v. We will remove this restriction, for we want to study the behaviour of energy tensor and linearised field for high velocities, which in the case of a rotating, hollow cylinder differs essentially from that of incoherent matter.

As an application we calculate the field of the thin, infinitely long, hollow

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‡ The author is indebted to the referee for drawing his attention to the work of G. L. Clark, who in another way has also investigated the gravitational field of rotating bodies (Clark, 1948, 1950).

cylinder, where the typical effects and analogies with electrodynamics are coming out particularly well.

In a second example we determine the stress tensor and the linearised field of a spinning rod (dumb-bell). We can show that our results, being valid in the short-range field as well as in the radiation zone, are in agreement with a far-field solution for small velocities given by Einstein (1916, 1918) and Landau & Lifshitz (1967a).†

2. *Energy Tensor of Uniformly Rotating Masses*

2.1. *Linearised Field Equations*

For the metric tensor $g_{\mu\nu}$ of space-time we make the ansatz‡

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \qquad \gamma_{\mu\nu} = \psi_{\mu\nu} - \tfrac{1}{2}\psi\eta_{\mu\nu}, \qquad \psi = \psi_{\mu\nu}\eta^{\mu\nu} \tag{2.1.1}$$

where $\eta_{\mu\nu}$ = Minkowski tensor (signature −2).

In order to take advantage of the symmetries of rotating bodies we do not restrict ourselves to rectangular coordinates. Under the condition§

$$\psi_{\mu}{}^{\nu}{}_{\|\nu} = 0 \tag{2.1.2}$$

the linearised field equations are

$$\psi_{\mu\nu}{}^{\|\kappa}{}_{\|\kappa} = -2\kappa T_{\mu\nu} \tag{2.1.3}$$

with

$$T_{\mu}{}^{\nu}{}_{\|\nu} = 0 \tag{2.1.4}$$

where $T_{\mu\nu}$ is the energy tensor without taking into account gravitational interactions. In Cartesian coordinates x, y, z we get the solution of (2.1.3) as retarded integrals:

$$\psi_{\mu\nu} = -\frac{\kappa}{2\pi}\int \frac{T_{\mu\nu}[x^a, t-(\bar{r}/c)]}{\bar{r}}\,dV \tag{2.1.5}$$

with dV as the three-dimensional volume element in the Minkowski space.

2.2. *Stress Tensor*

The total energy-momentum-stress tensor may be written in the well-known form (neglecting heat flow)

$$T_{\mu\nu} = \rho_0 c^2 u_\mu u_\nu - \sigma_{\mu\nu} \tag{2.2.1}$$

with the kinetic term $\rho_0 c^2 u_\mu u_\nu$ representing the motion and the stress term $\sigma_{\mu\nu}$ arising from the nongravitational interactions. ($\rho_0 c^2$: proper density

† The knowledge of the short-range field of the dumb-bell enables us to calculate the gravitational self-interaction of the system (radiation damping). These calculations will be published in a second paper.

‡ Greek suffices range and sum over 1, 2, 3, 4; Latin suffixes over 1, 2, 3.

§ $\|\alpha$ denotes covariant differentiation with respect to $\eta_{\mu\nu}$; $|\alpha$ partial differentiation.

of proper energy; u^μ: 4-velocity of matter). The symmetrical tensor $\sigma_{\mu\nu}$ has the following property (Synge, 1956):

$$\sigma_{\mu\nu} u^\nu = 0 \tag{2.2.2}$$

From the relations of orthogonality (2.2.2) and the conservation laws for $T^{\mu\nu}$ (2.1.4) it is easy to derive the equation of continuity for the '4-current of matter' in the case of rigid rotation:

$$(\rho_0 u^\nu)_{\|\nu} = 0 \tag{2.2.3}$$

Because of the symmetries of those systems, which we want to investigate, we introduce cylindrical coordinates r, ϕ, z. Then $\eta_{\mu\nu}$ takes the form

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{2.2.4}$$

In this coordinate system, for uniform rotation about the z-axis, the 4-velocity of any point of the body is given by

$$u^r = u^z = 0, \qquad u^\phi = \frac{\omega}{c\sqrt{(1-\beta^2)}}, \qquad u^4 = \frac{1}{\sqrt{(1-\beta^2)}} \tag{2.2.5}$$

$$\left(\omega = \frac{d\phi}{dt}, \qquad \beta = \frac{\omega r}{c}\right)$$

From the conservation equations (2.1.4) follows with regard to the relations (2.2.1), (2.2.2), (2.2.3) for $\sigma^{\mu\nu}$ in cylindrical coordinates†

$$\sigma^{r\nu}{}_{|\nu} - r\sigma^{\phi\phi} + \frac{1}{r}\sigma^{rr} = -\frac{\rho_0 \omega^2 r}{1-\beta^2}, \qquad \sigma^{\phi\nu}{}_{|\nu} + \frac{3}{r}\sigma^{r\phi} = 0,$$

$$\sigma^{z\nu}{}_{|\nu} + \frac{1}{r}\sigma^{zr} = 0, \qquad \sigma^{\mu\phi}_{;}\frac{\omega}{c}r^2 = \sigma^{\mu 4} \tag{2.2.6}$$

Any stress tensor of a uniformly rotating mass system, the rotation of which is induced by nongravitational forces, must be solution of (2.2.6) in the lowest order. But generally, the seven conditions (2.2.6) are not sufficient for the determination of all ten components of $\sigma^{\mu\nu}$ without knowing special elastic or symmetrical properties of the system.

3. *Gravitational Field of a Rotating Hollow Cylinder*

3.1. *Energy-Momentum-Stress Tensor*

In case the rotating body is a thin, hollow cylinder, only pure tangential stresses can be expected. Therefore, in cylindrical coordinates all space-like

† The eighth condition $\sigma^{4\mu}{}_{\|\mu} = 0$ is not independent of the other equations.

components of $\sigma^{\mu\nu}$ are zero, with the exception of $\sigma^{\phi\phi}$. Then $\sigma^{\mu\nu}$ is determined completely by (2.2.6) and one gets:

$$\sigma^{\phi\phi} = \frac{\rho_0\,\omega^2}{1-\beta^2}, \qquad \sigma^{\phi 4} = \sigma^{\phi\phi}\frac{\omega}{c}r^2, \qquad \sigma^{44} = \sigma^{\phi\phi}\frac{\omega^2}{c^2}r^4 \tag{3.1.1}$$

According to (2.2.1) and (2.2.5), the only non-vanishing components of the energy tensor are

$$T^{\phi 4} = T^{4\phi} = \rho_0\, c\omega, \qquad T^{44} = \rho_0\, c^2(1+\beta^2) \tag{3.1.2}$$

Transformation to Cartesian coordinates yields

$$T^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\rho_0\, c\omega y \\ 0 & 0 & 0 & \rho_0\, c\omega x \\ 0 & 0 & 0 & 0 \\ -\rho_0\, c\omega y & \rho_0\, c\omega x & 0 & \rho_0\, c^2(1+\beta^2) \end{pmatrix} \tag{3.1.3}$$

Thus we find the following. Firstly, in those components of $T^{\mu\nu}$, being space-like with reference to the inertial system, stress and kinetic tensor mutually cancel. Secondly, $T^{\mu\nu}$ depends on $\rho_0 c^2$ and not on $\rho_0 c^2/(1-\beta^2)$. So for high velocities the behaviour of $T^{\mu\nu}$ differs essentially from that of incoherent matter ($T_{\mu\nu} = \rho_0 c^2 u_\mu u_\nu$).

Restricting ourselves to elastic bodies, we see that the proper energy density $\rho_0 c^2 = T_{\mu\nu}\, u^\mu u^\nu$ is composed of two components:

$$\rho_0\, c^2 = \rho_{00}\, c^2 + \epsilon_0 \tag{3.1.4}$$

where the elastic energy ϵ_0 depends on rotation ω and elastic properties of the matter, while $\rho_{00} c^2$ is the proper energy density without rotation. From the theory of elasticity we know† (Landau & Lifshitz, 1967b) that in the case of small deformations the elastic energy is a quadratic function of the stress tensor. Therefore we should think that ϵ_0 behaves proportional to $\beta^4/(1-\beta^2)^2$. But this is not right at all. The only quadratic expressions which can be derived from $\sigma^{\mu\nu}$ are $\sigma_{\alpha\beta}\sigma^\alpha{}_\beta$ and $(\sigma_\alpha{}^\alpha)^2$. Then with (3.1.1) we easily find that

$$\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} = (\sigma_\alpha{}^\alpha)^2 = \rho_0{}^2\, c^4\, \beta^4 \tag{3.1.5a}$$

Thus

$$\epsilon_0 = \alpha\,.\,\rho_0{}^2\, c^4\, \beta^4 \tag{3.1.5b}$$

where α is a function of the elastic properties of the modulus of elasticity E and Poisson's ratio μ. If we know α, then from (3.1.4) and (3.1.5b) $\rho_0 c^2$ can be determined by a quadratic equation.

† In our case, for small deformations, the possibility of relativistic generalisation of classical 'deformation gradient' is obvious. Detailed discussions about general relativistic treatment of deformable bodies have recently be made [for example Bragg (1970) or Oldroyd (1970)].

3.2. *Integrals of Field Equations*

According to the well-known formula of transformation of three-dimensional volume

$$dV = dV_0\sqrt{(1-\beta^2)} \tag{3.2.1}$$

we substitute the V-integration in (2.1.4) by an integration over the proper space of matter V_0. Because of $\beta = \text{const.}$, the nonzero components of $\psi^{\mu\nu}$ are

$$\begin{aligned}
\psi^{x4} &= \frac{\kappa c}{2\pi}\omega\sqrt{(1-\beta^2)}\int\frac{\rho_0 y}{\bar{r}}dV_0 \\
\psi^{44} &= -\frac{\kappa c^2}{2\pi}(1+\beta^2)\sqrt{(1-\beta^2)}\int\frac{\rho_0}{\bar{r}}dV_0 \\
\psi^{y4} &= -\frac{\kappa c}{2\pi}\omega\sqrt{(1-\beta^2)}\int\frac{\rho_0 x}{\bar{r}}dV_0
\end{aligned} \tag{3.2.2}$$

We state, that in the (nonrealistic) case of vanishing elastic energy, i.e. rigidity ($\alpha = 0$), the integrals (3.2.2) are finite for each possible value of β and the gravitational field is becoming zero for $\beta \to 1$, with maxima lying at $\beta = \sqrt{\frac{1}{2}}$ for $\psi^{a4}(\beta)$ and at $\beta = \sqrt{\frac{1}{3}}$ for $\psi^{44}(\beta)$. Of course, real bodies will break up long before $\beta \approx 1$ is reached, and for all nonrigid bodies ($\alpha \neq 0$) there will be a limit in β, where the condition of small deformations is no longer valid. Nevertheless, it is interesting to note that for small, but only theoretical, values of α, the gravitational field $\psi^{\mu\nu}$, which is induced by the rotating hollow cylinder, may have maxima with respect to β before the body is breaking up.

3.3. *Infinitely Long, Hollow Cylinder*

Finally we determine the gravitational effects of a uniformly rotating, hollow cylinder, which is infinitely long and homogeneous. The explicit performance of integration of (3.2.2) yields for the nonzero components of the cylindrical symmetric field $\gamma_{\mu\nu}$ (2.1.1):

(1) $r > R$

$$\gamma_{x4} = -\frac{y}{x}\gamma_{y4} = 2\kappa c\omega\frac{R^2}{r^2}Fx,$$

$$\gamma_{\alpha\alpha} = \gamma_{44} = \tfrac{1}{2}\psi_{44} = [\kappa(1+\beta^2)c^2F]\ln\frac{r}{R} \tag{3.3.1a}$$

(2) $r < R$

$$\begin{aligned}
\gamma_{x4} &= -\kappa c\omega Fy, \\
\gamma_{y4} &= +\kappa c\omega Fx
\end{aligned} \tag{3.3.1b}$$

where $F = \int \rho_0 r\,dr$ and $R =$ radius of the cylinder.

Outside the cylinder we find a 'centrifugal' field but no 'coriolis' field, while inside there is a pure homogeneous 'coriolis' field. We note analogy to electrodynamics (infinitely long coil).

4. *Gravitational Field of a Spinning Rod (Dumb-bell)*

By the general conditions (2.2.6) for the stress tensor of rotating bodies not only systems with pure *tangential stresses* are marked out, but also those with *pure radial stresses*, where only $\sigma_r{}^r \neq 0$. Stresses of this kind are to be found, for example, in a thin rod rotating about its center of mass S. Then we get by integration of (2.2.6) (in cylindrical coordinates):

$$\sigma^{rr} = -\frac{1}{r}\left[\int_0^r \rho\omega^2 r_1{}^2\, dr_1 + A\right], \qquad \rho = \frac{\rho_0}{1-\beta^2} \tag{4.1a}$$

$\sigma^{\mu\nu} = 0$ for all other components. The constant of integration A is determined by the condition, that σ^{rr} vanishes at the ends of the rod. In case S lies in the center point of the rod (length $= 2R$), one gets

$$A = -\int_0^R \rho\omega^2 r_1{}^2\, dr_1 \tag{4.1b}$$

Herewith the energy tensor is in Cartesian coordinates:

$$T^{\mu\nu} = \begin{pmatrix} \rho\omega^2 y^2 - \frac{x^2}{r^2}\sigma^{rr} & -\rho\omega^2 xy - \frac{xy}{r^2}\sigma^{rr} & 0 & -\rho c\omega y \\ -\rho\omega^2 xy - \frac{xy}{r^2}\sigma^{rr} & \rho\omega^2 x^2 - \frac{y^2}{r^2}\sigma^{rr} & 0 & \rho c\omega x \\ 0 & 0 & 0 & 0 \\ -\rho c\omega y & \rho c\omega x & 0 & \rho c^2 \end{pmatrix} \tag{4.2}$$

As in the case of incoherent matter energy-density with reference to the inertial system is of the form $\rho c^2 = \rho_0 c^2/(1-\beta^2)$ in opposition to that of the rotating hollow cylinder (pure tangential stresses), compare (3.1.3).

The gravitational field, belonging to (4.2), is given by the retarded integrals (2.2.6). Specialising to a dumb-bell, the mass of the rod may be neglected, compared with the two equal masses at the ends (proper masses m_0). Of course this is valid only as long as elastic energy within the rod is small (see Section 3). Then the integrals (2.2.6) may be calculated by the method of Lienard-Wiechert, and one finds:

$$\psi^{xx} = -E\beta^2\left\{\sum_{i=1,2}\frac{\sin^2\omega t'_{(i)}}{\bar{R}'_{(i)}} - \frac{1}{R}\int_0^{R*} dr\frac{\cos^2\omega t'_{(r)}}{\bar{R}'_{(r)}}\right\}$$

$$\psi^{yy} = -E\beta^2\left\{\sum_{i=1,2}\frac{\cos^2\omega t'_{(i)}}{\bar{R}'_{(i)}} - \frac{1}{R}\int_0^{R*} dr\frac{\sin^2\omega t'_{(r)}}{\bar{R}'_{(r)}}\right\}$$

$$\psi^{xy} = E\beta^2 \left\{ \sum_{i=1,2} \frac{\cos\omega t'_{(i)} \sin\omega t'_{(i)}}{\bar{R}'_{(i)}} + \frac{1}{R} \int_0^{R} {}^{*}\, dr \frac{\cos\omega t'_{(i)} \sin\omega t'_{(i)}}{\bar{R}'_{(r)}} \right\}$$

$$\psi^{x4} = -E\beta \sum_{i=1,2} \frac{(-1)^i \sin\omega t'_{(i)}}{\bar{R}'_{(i)}}$$

$$\psi^{y4} = E\beta \sum_{i=1,2} \frac{(-1)^i \cos\omega t'_{(i)}}{\bar{R}'_{(i)}}$$

$$\psi^{44} = -E \sum_{i=1,2} \frac{1}{\bar{R}'_{(i)}}$$

$$E = \frac{\kappa c^2}{2\pi} \frac{m_0}{\sqrt{(1-\beta^2)}}, \qquad t' = t - \frac{\bar{r}'}{c}, \qquad \bar{r}' = \bar{r}(t'), \qquad \bar{R}' = \bar{R}(t')$$

$$\bar{R} = \bar{r} \pm \beta(x \sin\omega t - y\cos\omega t) \tag{4.3}$$

where + and − stand for points on the right and left side of the rod, respectively; (1) and (2) denote the first and second mass, (r) a point of the rod; and * upon the integrals means that integration over the left and right side of the rod is to be taken separately with regard to the retardation.

As is well known, Einstein has given a far-field approximation for $\beta^2 \ll 1$ [compare Einstein (1916, 1918) or Landau & Lifshitz (1967a)],

$$\psi^{ab} = -\frac{\kappa}{4\pi} \frac{1}{r_0} \frac{\partial^2}{\partial t^2} \int \rho_{00}\, x^a x^b \, dV_{[t - r_0/c]} \tag{4.4}$$

where r_0 is the distance from the mass-system to the field point. The components $\psi^{\mu 4}$ are determined by (2.1.2). We easily see that our result (4.3), being valid in the short-range field as well as in the radiation zone and without restriction upon β, is in agreement with (4.4).

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References

Bass, L. and Pirani, F. A. E. (1955). *Philosophical Magazine*, **46**, 850.
Bragg, L. E. (1970). *Journal of Mathematical Physics*, **11**, 1.
Brill, D. R. and Cohen, J. M. (1965). *Physical Review*, **143**, 527.
Clark, G. L. (1948). *Philosophical Magazine*, **39**, 747.
Clark, G. L. (1950). *Proceedings of the Royal Society*, **A201**, 488.
Einstein, A. (1916). *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 688.
Einstein, A. (1918). *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 155.
Hönl, H. and Maue, A. W. (1956). *Zeitschrift für Physik*, **144**, 152.

Landau, L. D. and Lifschitz, E. M. (1967a). *Klassische Feldtheorie*. Akademie-Verlag, Berlin.
Landau, L. D. and Lifschitz, E. M. (1967b). *Elastizitätstheorie*. Akademie-Verlag, Berlin.
Lense, J. and Thirring, H. (1918). *Physikalische Zeitschrift*, **19**, 156.
Oldroyd, J. G. (1970). *Proceedings of the Royal Society*, **A316**, 1.
Synge, J. L. (1956). *Relativity, The Special Theory*. North-Holland Publishing Company, Amsterdam.
Teyssandier, P. (1970). *Annales Institut H. Poincaré*, **12**, 263.
Thirring, H. (1918). *Physikalische Zeitschrift*, **19**, 33.
Thirring, H. (1921). *Physikalische Zeitschrift*, **22**, 29.